

MIXING ON RANK-ONE TRANSFORMATIONS

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ABSTRACT. We prove that mixing on rank-one transformations is equivalent to the spacer sequence being slice-ergodic. Slice-ergodicity, introduced in this paper, generalizes the notion of ergodic sequence to the uniform convergence of ergodic averages (as in the mean ergodic theorem) over subsequences of partial sums. We show that polynomial staircase transformations satisfy this condition and therefore are mixing.

1. INTRODUCTION

1.1. Rank-One Transformations. Rank-one transformations are transformations “well-approximated” by a sequence of discrete spectrum transformations, so it was very surprising when in 1970 Ornstein [Orn72] showed the existence of rank-one *mixing* transformations. Rank-one mixing transformations are mixing of all orders [Kal84], [Rhy93] and enjoy other remarkable properties, see e.g. [Kin88]. Ornstein’s construction was stochastic in nature: there is a class of rank-one transformations so that almost surely a transformation in that class is mixing; however, it did not yield a deterministic procedure for constructing one.

1.2. Staircase Transformations. A few years later, Smorodinsky conjectured that a specific rank-one transformation, the classic staircase transformation, is mixing. In 1992, Adams and Friedman [AF92] gave a deterministic algorithm involving a sequence of cutting and stacking constructions that produced a mixing rank-one transformation, and later Adams [Ada98] proved that Smorodinsky’s conjecture is true. Informally, a staircase transformation is a cutting and stacking transformation with sequence $\{r_n\}$ of natural numbers such that at the n^{th} stage the n^{th} column or stack is cut into r_n subcolumns and “spacers” (see Section 5) are placed in a staircase fashion on the subcolumns before stacking, i.e., the number of spacers in each subsequent subcolumn is increased by 1. Adams showed that the resulting staircase transformation is mixing provided that $\frac{r_n^2}{h_n} \rightarrow 0$ as $n \rightarrow \infty$ (which also implies that T is finite measure-preserving), where h_n denotes the number of levels, or height, of the n^{th} column. He then asked whether the mixing property holds for every finite measure-preserving staircase transformation simply under the assumption that $r_n \rightarrow \infty$. In 2003, Ryzhikov wrote the authors a short email stating that in 2000 he gave a lecture where he proved that all staircases are mixing [Rhy03] (giving a positive answer to Adams’ question), but no argument was included and no preprint has been available. After this paper was completed, Ryzhikov informed the authors that his paper was forthcoming. We would also like to thank Ryzhikov for asking a question that clarified our writing of Section 9. The application of our

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main theorem shows that polynomial staircase transformations are mixing (Theorem 10). Specializing to the case of linear polynomials gives another proof that staircase transformations are mixing.

1.3. Restricted Growth. The $\frac{r_n^2}{h_n} \rightarrow 0$ condition, a restriction on the asymptotic growth of the spacers relative to the column height, was generalized to all rank-one transformations and called “restricted growth” in [CS04]. The staircase transformation of Smorodinsky’s conjecture is obtained when $r_n = n + 1$; verifying that it satisfies the restricted growth condition is straightforward. In [CS04], the authors proved an equivalence between mixing and a condition on the spacer sequence for rank-one transformations with restricted growth. It followed that restricted growth rank-one transformations with the sequence of spacers given by a polynomial satisfying some general conditions (including the staircases of [Ada98]) are mixing. Ornstein’s result also follows from that theorem.

1.4. Our Result. In this paper we lift the restricted growth condition from the theorems in [CS04]. We introduce the notion of a slice-ergodic sequence and prove in Theorem 4 that a rank-one transformation is mixing if and only if its spacer sequence is slice-ergodic. We use this theorem to show that all polynomial staircase transformations are mixing.

2. MIXING PROPERTIES

2.1. Dynamical Systems. For our study, **dynamical system** shall mean a standard probability measure space (X, \mathcal{B}, μ) and **transformation** $T : X \rightarrow X$ that is invertible, measurable and measure-preserving. Throughout the paper, $X = [0, 1)$, μ is Lebesgue measure on X and \mathcal{B} is the algebra of μ -measurable subsets of X .

2.2. Mixing. A transformation T is **mixing** when for all $A, B \in \mathcal{B}$,

$$\lim_{n \rightarrow \infty} \mu(T^n(A) \cap B) - \mu(A)\mu(B) = 0;$$

$\{t_n\}$ is a **mixing sequence** (with respect to T) when for all $A, B \in \mathcal{B}$,

$$\lim_{n \rightarrow \infty} \mu(T^{t_n}(A) \cap B) - \mu(A)\mu(B) = 0.$$

2.3. Ergodicity. A transformation T is **ergodic** when for all $A \in \mathcal{B}$, if $T^{-1}(A) = A$ then $\mu(A) = 0$ or $\mu(A) = 1$. The **mean (von Neumann) ergodic theorem** states that T is ergodic if and only if for all $B \in \mathcal{B}$,

$$\lim_{n \rightarrow \infty} \int \left| \frac{1}{n} \sum_{j=0}^{n-1} \chi_B \circ T^{-j} - \mu(B) \right| d\mu = 0.$$

(χ_B , the characteristic function of the set B .) A transformation T is **totally ergodic** when for any $\ell \in \mathbb{N}^+$, the transformation T^ℓ is ergodic. (We use the notation $\mathbb{N} = \{0, 1, 2, \dots\}$ for the natural numbers; $\mathbb{N}^+ = \{1, 2, \dots\}$ for the positive natural numbers; and $\mathbb{Z}_N = \{0, 1, \dots, N-1\}$ for the N -element additive group).

2.4. Ergodic Sequences. The term **sequence** shall mean sequence in \mathbb{N} that is strictly increasing. A sequence $\{a_n\}$ is an **ergodic sequence** (with respect to a

transformation T) when for all $B \in \mathcal{B}$,

$$\lim_{n \rightarrow \infty} \int \left| \frac{1}{n} \sum_{j=0}^{n-1} \chi_B \circ T^{-a_j} - \mu(B) \right| d\mu = 0.$$

3. DYNAMICAL SEQUENCES

3.1. Basic Notions. Dynamical sequences were introduced in [CS04].

Definition 3.1. A dynamical sequence $\{s_{n,j}\}_{\{r_n\}}$ is a doubly-indexed collection of integers $s_{n,j}$ for $n \in \mathbb{N}$ and $j \in \mathbb{Z}_{r_n}$ where $\{r_n\}$ is a given sequence, called the **index sequence**, which must have the property that $\lim_{n \rightarrow \infty} r_n = \infty$ (see [CS04]). The integer $s_{n,j}$ is the j^{th} **element** of the dynamical sequence at the n^{th} stage.

3.2. Partial Sums of Dynamical Sequences.

Notation. Let $\{s_{n,j}\}_{\{r_n\}}$ be a dynamical sequence and $n \in \mathbb{N}$, $j \in \mathbb{Z}_{r_n}$, $k \in \mathbb{Z}_{r_n-j}$. The k^{th} partial sum of the j^{th} element at the n^{th} stage is

$$s_{n,j}^{(k)} = \sum_{z=0}^{k-1} s_{n,j+z}.$$

Definition 3.2. Let $\{s_{n,j}\}_{\{r_n\}}$ be a dynamical sequence and $k \in \mathbb{N}$. The k^{th} **partial sum dynamical sequence** is the dynamical sequence $\{s_{n,j}^{(k)}\}_{\{r_n-k\}}$ (n “begins” at the smallest value such that $r_n \geq k$) whose elements are the k^{th} partial sums of $\{s_{n,j}\}_{\{r_n\}}$. Let $\{k_n\}$ be a sequence such that $k_n < r_n$ for all n . The $\{k_n\}^{\text{th}}$ **partial sum dynamical sequence** of $\{s_{n,j}\}_{\{r_n\}}$ is the dynamical sequence $\{s_{n,j}^{(k_n)}\}_{\{r_n-k_n\}}$.

3.3. Monotonic Dynamical Sequences.

Definition 3.3. A dynamical sequence $\{s_{n,j}\}_{\{r_n\}}$ is **monotone** when for every fixed $M \in \mathbb{N}$, $\lim_{n \rightarrow \infty} \frac{1}{r_n} \# \{j \in \mathbb{Z}_{r_n} : |s_{n,j}| < M\} = 0$ (the symbol $\#$ denotes cardinality).

3.4. Slicings of Dynamical Sequences.

Definition 3.4. Let $\{s_{n,j}\}_{\{r_n\}}$ be a dynamical sequence and let $\{Q_n\}$ be a sequence such that $\lim_{n \rightarrow \infty} \frac{Q_n}{r_n} = 0$. A collection of sets $\Gamma_{n,q} \subseteq \mathbb{Z}_{r_n}$ and maps $\Psi_{n,q} : \mathbb{Z}_{r_n} \rightarrow \mathbb{Z}$ for $n \in \mathbb{N}$ and $q \in \mathbb{Z}_{Q_n}$ is a **slicing** of $\{s_{n,j}\}_{\{r_n\}}$ when for each $n \in \mathbb{N}$ the $\Gamma_{n,q}$ partition \mathbb{Z}_{r_n} (recall that $\{\Gamma_q\}$ partition \mathbb{Z}_r when $\bigcup \Gamma_q = \mathbb{Z}_r$ and $\Gamma_q \cap \Gamma_{q'} = \emptyset$ for $q \neq q'$), and for each $q \in \mathbb{Z}_{Q_n}$ there exists $a_{n,q}, b_{n,q} \in \mathbb{N}$ such that for all $j \in \mathbb{Z}_{r_n}$, if $a_{n,q} \leq s_{n,j} < b_{n,q}$ then $j \in \Gamma_{n,q}$.

Definition 3.5. Let $\{s_{n,j}\}_{\{r_n\}}$ be a dynamical sequence and $\{k_n\}$ a sequence such that $k_n < r_n$ for all $n \in \mathbb{N}$. Let $\{\Gamma_{n,q}\}$ and $\{\Psi_{n,q}\}$ indexed over $\{Q_n\}$ be a slicing of the partial sum dynamical sequence $\{s_{n,j}^{(k_n)}\}_{\{r_n-k_n\}}$ and let $\{\alpha_{n,q}\}_{\{Q_n\}}$ be a dynamical sequence such that $\alpha_{n,q} < k_n$ for all $n \in \mathbb{N}$ and $q \in \mathbb{Z}_{r_n-k_n}$. The dynamical sequences $\{s_{n,\Psi_{n,q}(j)}^{(k_n-\alpha_{n,q})}\}_{\{\#\Gamma_{n,q}\}}$ are an **approximate slicing** of the $\{k_n\}^{\text{th}}$ partial sum dynamical sequence of $\{s_{n,j}\}_{\{r_n\}}$.

4. ERGODICITY ON DYNAMICAL SEQUENCES

4.1. Ergodic Dynamical Sequences.

Definition 4.1. A dynamical sequence $\{s_{n,j}\}_{\{r_n\}}$ is an **ergodic dynamical sequence** (with respect to a transformation T) when for all $B \in \mathcal{B}$,

$$\lim_{n \rightarrow \infty} \int \left| \frac{1}{r_n} \sum_{j=0}^{r_n-1} \chi_B \circ T^{-s_{n,j}} - \mu(B) \right| d\mu = 0.$$

4.2. Slice-Ergodicity.

Definition 4.2. Let $\{s_{n,j}\}_{\{r_n\}}$ be a dynamical sequence and $\{k_n\}$ be a sequence such that $k_n < r_n$ for all $n \in \mathbb{N}$. Then $\{s_{n,j}\}_{\{r_n\}}$ is **slice-ergodic around** $\{k_n\}$ (with respect to a transformation T) if for every approximate slicing of the dynamical sequence $\{s_{n,j}^{(k_n)}\}_{\{r_n - k_n\}}$ defined by sets $\{\Gamma_{n,q}\}$, maps $\{\Psi_{n,q}\}$, sequence $\{Q_n\}$, and dynamical sequence $\{\alpha_{n,q}\}_{\{Q_n\}}$, and all $B \in \mathcal{B}$,

$$\lim_{n \rightarrow \infty} \int \left| \frac{1}{r_{p_n}} \sum_{q=0}^{Q_n-1} \sum_{j \in \Gamma_{n,q}} \chi_B \circ T^{-s_{n,j}^{(k_n - \alpha_{n,q})}} - \mu(B) \right| d\mu = 0,$$

i.e., the ergodic average over the approximate slicing tends to zero. A dynamical sequence is **slice-ergodic** when it is slice-ergodic around every sequence $\{k_n\}$ such that $k_n < r_n$ for all $n \in \mathbb{N}$.

4.3. Mixing and Ergodic Dynamical Sequences. The following standard generalization of the Blum-Hanson theorem from sequences to dynamical sequences is shown in [CS04].

Theorem 1. A transformation T is mixing if and only if every monotone dynamical sequence is ergodic with respect to T .

5. RANK-ONE TRANSFORMATIONS

5.1. Cutting and Stacking. Begin with $[0, 1)$, the only “level” in the initial “column”. “Cut” it into r_0 “sublevels”, pieces of equal length: $[0, \frac{1}{r_0})$, $[\frac{1}{r_0}, \frac{2}{r_0})$, \dots , $[\frac{r_0-1}{r_0}, 1)$. Place $s_{0,0}$ intervals of the same length “above” $[0, \frac{1}{r_0})$, i.e., if $s_{0,0} = 1$ place $[1, \frac{r_0+1}{r_0})$ above $[0, \frac{1}{r_0})$. Likewise, place $s_{0,j}$ “spacer” sublevels above each piece. Now, “stack” the resulting subcolumns from left to right by placing $[0, \frac{1}{r_0})$ at the bottom, the $s_{0,0}$ spacers above it, $[\frac{1}{r_0}, \frac{2}{r_0})$ above the topmost of the $s_{0,0}$ spacers and so on, ending with the topmost of the s_{0,r_0-1} spacers. This stack of $h_1 = r_0 + \sum_{j=0}^{r_0-1} s_{0,j}$ levels (of length $\frac{1}{r_0}$), the second column, defines a map $T_0 : [0, 1 + \frac{1}{r_0} \sum_{j=0}^{r_0-1} s_{0,j} - \frac{1}{r_0}) \rightarrow [\frac{1}{r_0}, 1 + \frac{1}{r_0} \sum_{j=0}^{r_0-1} s_{0,j})$ that sends points directly up one level.

Repeat the process: cut the entire new column into r_1 subcolumns of equal width $\frac{1}{r_0 r_1}$, preserving the stack map on each subcolumn; place $s_{1,j}$ spacers (intervals not yet in the space the same width as the subcolumns) above each subcolumn ($j \in \mathbb{Z}_{r_1}$); and stack the resulting subcolumns from left to right. Our new column defines a map T_1 that agrees with T_0 where it is defined and extends it to all but the topmost spacer of the rightmost subcolumn. Iterating this process leads to a transformation T defined on all but a Lebesgue measure zero set.

5.2. Construction of Rank-One Transformations. A transformation created by *cutting and stacking* as just described (with a single column resulting from each iteration) is a **rank-one transformation**. The reader is referred to [Fer97] and [Fri70] for more details. Rank-one transformations are measurable and measure-preserving under Lebesgue measure, and are completely defined by a dynamical sequence $\{s_{n,j}\}_{\{r_n\}}$ where at the n^{th} step we cut into r_n pieces and place $s_{n,j}$ spacers above each subcolumn. This $\{s_{n,j}\}_{\{r_n\}}$ is the **spacer sequence** for the transformation and $\{r_n\}$ is the **cut sequence**. The **height sequence** $\{h_n\}$ is the number of levels in each column: $h_0 = 1$ and $h_{n+1} = r_n h_n + \sum_{j=0}^{r_n-1} s_{n,j}$.

We write $I_{n,i}$ to denote the i^{th} level in the n^{th} stack ($i \in \mathbb{Z}_{h_n}$) where $I_{n,0}$ is the bottom level and $T(I_{n,i}) = I_{n,i+1}$ and write $C_n = \bigcup_{i=0}^{h_n-1} I_{n,i}$ to denote the n^{th} column and $S_n = C_{n+1} \setminus C_n$ to denote the spacers added. We write $I_{n,i}^{[j]}$ for the j^{th} sublevel of the i^{th} level of the n^{th} column, i.e., $I_{n,0}^{[0]}$ is the leftmost sublevel of the bottom level ($I_{n,0}^{[0]} = I_{n+1,0}$ becomes the bottom level of the next column). Note that T is defined on a finite measure space if and only if $\sum_{n=0}^{\infty} \mu(S_n) < \infty$ and in that case T is isomorphic to the transformation defined on $[0, 1)$ obtained by cutting and stacking in the same fashion as T but beginning with $C_0 = [0, \frac{1}{K})$ where K is the measure of the space the original T is defined on.

5.3. Rank-One Uniform Mixing. Rank-one uniform mixing involves sums of mixing values over increasingly fine levels. Introduced in [CS04], details and proofs may be found there.

Definition 5.1. Let T be a rank-one transformation with heights $\{h_n\}$ and levels $\{I_{n,i}\}$ and $\{a_n\}$ a sequence. Set p_n such that $h_{p_n} \leq a_n < h_{p_n+1}$. Then $\{a_n\}$ is **rank-one uniform mixing** (with respect to T) when for all $B \in \mathcal{B}$,

$$\lim_{n \rightarrow \infty} \sum_{i=0}^{h_{p_n}-1} |\mu(T^{a_n}(I_{p_n,i}) \cap B) - \mu(I_{p_n,i})\mu(B)| = 0;$$

T is **rank-one uniform mixing** when \mathbb{N} is a rank-one uniform mixing sequence (with respect to T).

Proposition 5.1. [CS04] Let T be a rank-one transformation. If a sequence $\{t_n\}$ is rank-one uniform mixing (with respect to T) then $\{t_n\}$ is mixing (with respect to T). Consequently, if T is rank-one uniform mixing then T is mixing.

5.4. Levels of Rank-One Transformations.

Lemma 5.1. Let T be a rank-one transformation with levels $\{I_{n,i}\}$, heights $\{h_n\}$, and spacers $\{s_{n,j}\}_{\{r_n\}}$. Let $p \in \mathbb{N}$, $i \in \mathbb{Z}_{h_p}$, $j \in \mathbb{Z}_{r_p}$, $k \in \mathbb{Z}_{r_p-j}$, $t \in \mathbb{Z}_{h_p-i}$ and B a union of levels in C_p . Then the following hold:

- (i) $I_{p,i} = \bigcup_{j=0}^{r_p-1} I_{p,i}^{[j]}$;
- (ii) $T^{kh_p+s_{p,j}^{(k)}}(I_{p,i}^{[j]}) = I_{p,i}^{[j+k]}$; and
- (iii) $\mu(T^t(I_{p,i}^{[j]}) \cap B) = \frac{1}{r_p} \mu(T^t(I_{p,i}) \cap B)$.

Proof. (i) and (ii) follow from the construction of rank-one transformations. For (iii), $T^t(I_{p,i}^{[j]}) = I_{p,i+t}^{[j]}$ and $B \subseteq C_p$ so $I_{p,i+t} \subseteq B$ or $I_{p,i+t} \cap B = \emptyset$. \square

Lemma 5.2. *For any $p \in \mathbb{N}$, any $\Lambda \subseteq \mathbb{Z}_{h_N}$, any $\Gamma \subseteq \mathbb{N}$, any B a union of levels in C_p and any maps $f : \Gamma \rightarrow \mathbb{Z}$ and $g : \Gamma \rightarrow \mathbb{Z}_{r_p}$,*

$$\begin{aligned} & \sum_{i \in \Lambda} \left| \sum_{j \in \Gamma} \mu(T^{f(j)}(I_{p,i}^{[g(j)]}) \cap B) - \mu(I_{p,i}^{[g(j)]})\mu(B) \right| \\ & \leq \int \left| \frac{1}{r_p} \sum_{j \in \Gamma} \chi_B \circ T^{f(j)} - \mu(B) \right| d\mu + \left(\sup_{j \in \Gamma} f(j) \right) \frac{1}{h_p} \frac{\#\Gamma}{r_p}. \end{aligned}$$

Proof. Lemma 5.1 (iii) and the definition of integration. \square

6. MIXING SEQUENCE THEOREMS

6.1. Mixing Height Sequences.

Theorem 2. *Let T be a rank-one transformation with spacers $\{s_{n,j}\}_{\{r_n\}}$ and heights $\{h_n\}$ and let $k \in \mathbb{N}$. If $\{s_{n,j}^{(k)}\}_{\{r_{n-k}\}}$ is ergodic (with respect to T) then $\{kh_n\}$ is rank-one uniform mixing (with respect to T).*

Proof. Let T , $\{s_{n,j}\}_{\{r_n\}}$, $\{h_n\}$ and k be as above. Let B be a union of levels in C_N for some fixed $N \in \mathbb{N}$ (note that levels approximate measurable sets). For any sets $J_n \in \mathbb{Z}_{r_n}$, apply Lemmas 5.1 (i) then 5.1 (ii) and finally Lemma 5.2, for any $n \geq N$,

$$\begin{aligned} & \sum_{i=0}^{h_n-1} \left| \mu(T^{kh_n}(I_{n,i}) \cap B) - \mu(I_{n,i})\mu(B) \right| \\ & \leq \sum_{i=0}^{h_n-1} \left| \sum_{j \in \mathbb{Z}_{r_n-k} \setminus J_n} \mu(T^{kh_n}(I_{n,i}^{[j]}) \cap B) - \mu(I_{n,i}^{[j]})\mu(B) \right| + \frac{k}{r_n} + \frac{\#J_n}{r_n} \\ & = \sum_{i=0}^{h_n-1} \left| \sum_{j \in \mathbb{Z}_{r_n-k} \setminus J_n} \mu(T^{s_{n,j}^{(k)}}(I_{n,i}^{[j+k]}) \cap B) - \mu(I_{n,i}^{[j+k]})\mu(B) \right| + \frac{k}{r_n} + \frac{\#J_n}{r_n} \\ & \leq \int \left| \frac{1}{r_n} \sum_{j=0}^{r_n} \chi_B \circ T^{s_{n,j}^{(k)}} - \mu(B) \right| d\mu + 2\frac{k}{r_n} + 2\frac{\#J_n}{r_n} + \frac{1}{h_n} \sup_{j \in \mathbb{Z}_{r_n-k} \setminus J_n} s_{n,j}^{(k)}. \end{aligned}$$

As k is fixed and $\{s_{n,j}^{(k)}\}_{\{r_{n-k}\}}$ is ergodic with respect to T , we need only show that there exists sets $J_n \subseteq \mathbb{Z}_{r_n}$ such that $\frac{\#J_n}{r_n} \rightarrow 0$ and $\frac{1}{h_n} \sup_{j \notin J_n} s_{n,j}^{(k)} \rightarrow 0$. Suppose not. Then there exists $\delta > 0$ such that $s_{n,j}^{(k)} \geq \delta h_n$ for at least δr_n values of j (for infinitely many n). But then at least $\frac{1}{k}\delta r_n$ values of j are such that $s_{n,j} \geq \frac{\delta}{k}$ so $\mu(S_n) \geq \frac{\delta^2}{k^2} r_n h_n \mu(I_{n+1,0}) = \frac{\delta^2}{k^2} \mu(C_n)$ contradicting that T is defined on a finite measure space. \square

6.2. Mixing Sequences.

Theorem 3. *Let T be a rank-one transformation with spacers $\{s_{n,j}\}_{\{r_n\}}$ and heights $\{h_n\}$ and let $\{t_n\}$ be a sequence. For each $n \in \mathbb{N}$, set p_n (uniquely) such that $h_{p_n} \leq t_n < h_{p_n+1}$ and set k_n (uniquely) such that $k_n h_{p_n} \leq t_n < (k_n + 1) h_{p_n}$.*

If $\{s_{n,j}\}_{\{r_n\}}$ is slice-ergodic around $\{k_n + 1\}$ then the sequence $\{t_n\}$ is rank-one uniform mixing (with respect to T).

Proof. We begin with a brief outline of the method undertaken. First, we dispose of the case when $\frac{k_n}{r_{p_n}} \rightarrow 1$ as it is trivial from the preceding theorem. We begin by slicing the spacer sequence into blocks of values with difference less than $\epsilon_n h_{p_n}$ ($\epsilon_n \rightarrow 0$), forming Q_n slices at each stage. Next, we determine $\alpha_{n,q}$, the number of times each subcolumn in the q^{th} block will be mapped through the top of the stack under t_n . Then we show which sublevel each sublevel is mapped to under T^{t_n} and or each of the three cases arising, and for each q , we show that the rank-one uniform mixing sum is small. The proof is completed by showing the combined sum of the three cases over all the q tends to zero by the slice-ergodicity of the spacers.

Let T , $\{s_{n,j}\}_{\{r_n\}}$, $\{h_n\}$, $\{t_n\}$, $\{p_n\}$, and $\{k_n\}$ be as above. Let $\{C_n\}$, $\{S_n\}$, and $\{I_{n,i}\}$ be the columns, spacers, and levels, for T , respectively. Let B be a union of levels in C_N for some $N \in \mathbb{N}$. Let $n \in \mathbb{N}$ such that $t_n \geq h_N$. Set $m_n = t_n - k_n h_{p_n}$ so $m_n \in \mathbb{Z}_{h_{p_n}}$.

For n such that $\frac{k_n}{r_{p_n}} \rightarrow 1$, apply Lemma 5.1 (i) and the triangle inequality,

$$\begin{aligned} & \sum_{i=0}^{h_{p_n}-1} |\mu(T^{t_n}(I_{p_n,i}) \cap B) - \mu(I_{p_n,i})\mu(B)| \\ & \leq \sum_{i=0}^{h_{p_n}-1} \sum_{j=0}^{r_{p_n}-1} |\mu(T^{k_n h_{p_n} + m_n}(I_{p_n,i}^{[j]}) \cap B) - \mu(I_{p_n,i}^{[j]})\mu(B)| \\ & \leq \sum_{i=0}^{h_{p_n+1}-1} |\mu(T^{h_{p_n+1}}(I_{p_n+1,i}) \cap B) - \mu(I_{p_n+1,i})\mu(B)| \\ & \quad + (h_{p_n+1} - k_n h_{p_n} - m_n)\mu(I_{p_n+1,0}) \end{aligned}$$

as there are at most $(h_{p_n+1} - k_n h_{p_n} - m_n)$ sublevels that do not “map through” the top of C_{p_n+1} . This quantity approaches zero as $n \rightarrow \infty$ since $\{h_n\}$ is rank-one uniform mixing with respect to T by Theorem 2 and since $k_n \approx r_{p_n}$,

$$\begin{aligned} (h_{p_n+1} - k_n h_{p_n} - m_n)\mu(I_{p_n+1,0}) & \leq (h_{p_n+1} - k_n h_{p_n})\mu(I_{p_n+1,0}) \\ & \approx (h_{p_n+1} - r_{p_n} h_{p_n})\mu(I_{p_n+1,0}) = \mu(S_{p_n}) \end{aligned}$$

and $\mu(S_{p_n}) \rightarrow 0$ as $n \rightarrow \infty$ because the final space has finite total measure.

Now consider when $\frac{k_n}{r_{p_n}}$ is bounded away from 1. We define the sequence $\{\epsilon_n\}$ by choosing $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$ such that $\frac{1}{\epsilon_n} \frac{\mu(S_n)}{\mu(C_n)} \rightarrow 0$ as $n \rightarrow \infty$ (possible as $\frac{\mu(S_n)}{\mu(C_n)} \rightarrow 0$ since the final space has finite total measure).

Let $\Psi_n : \mathbb{Z}_{r_{p_n}-k_n} \rightarrow \mathbb{Z}_{r_{p_n}-k_n}$ be a map such that $s_{p_n, \Psi_n(j)}^{(k_n)} \leq s_{p_n, \Psi_n(j+1)}^{(k_n)}$ for all $j \in \mathbb{Z}_{r_{p_n}-k_n-1}$. Set $\ell_{n,0} = 0$ and $\alpha_{n,0} = 1$ and then proceed inductively to define $\ell_{n,q+1}$ and $\alpha_{n,q+1}$ given $\ell_{n,q}$ and $\alpha_{n,q}$ as follows: choose $\ell_{n,q+1}$ to be the smallest positive integer less than $r_{p_n} - k_n$ such that

$$s_{p_n, \Psi_n(\ell_{n,q+1})}^{(k_n - \alpha_{n,q} + 1)} - s_{p_n, \Psi_n(\ell_{n,q})}^{(k_n - \alpha_{n,q})} \geq \epsilon_n h_{p_n}$$

if such an integer exists and choose $\ell_{n,q+1} = r_{p_n}$ and set $Q_n = q + 1$ if not. Choose $\alpha_{n,q+1}$ such that

$$\text{i) } (\alpha_{n,q} - 1)h_{p_n} + s_{p_n, \Psi_n(\ell_{n,q}) + k_n - \alpha_{n,q} + 1}^{(\alpha_{n,q} - 1)} < s_{p_n, \Psi_n(\ell_{n,q})}^{(k_n)} - m_n; \text{ and}$$

$$\text{ii)} \quad s_{p_n, \Psi_n(\ell_{n,q})}^{(k_n)} - m_n \leq \alpha_{n,q} h_{p_n} + s_{p_n, \Psi_n(\ell_{n,q}) + k_n - \alpha_{n,q}}^{(\alpha_{n,q})}.$$

Set $\beta_{n,q} = \alpha_{n,q} h_{p_n} + s_{p_n, \Psi_n(\ell_{n,q}) + k_n - \alpha_{n,q}}^{(\alpha_{n,q})} - s_{p_n, \Psi_n(\ell_{n,q})}^{(k_n)} + m_n$ and note that $0 \leq \beta_{n,q} < h_{p_n} + s_{p_n, \Psi_n(\ell_{n,q}) + k_n - \alpha_{n,q}}$. Set $\beta'_{n,q} = \beta_{n,q} - s_{p_n, \Psi_n(\ell_{n,q}) + k_n - \alpha_{n,q}}$ and note that $\beta'_{n,q} < h_{p_n}$.

For all $q \in \mathbb{Z}_{Q_n}$, define the sets

$$\Gamma_{n,q} = \{j \in \mathbb{Z}_{r_{p_n}} : s_{p_n, \Psi_n(\ell_{n,q})}^{(k_n)} \leq s_{p_n, j}^{(k_n)} < s_{p_n, \Psi_n(\ell_{n,q+1})}^{(k_n)}\}$$

and the maps $\Psi_{n,q} : \mathbb{Z}_{\ell_{n,q+1} - \ell_{n,q}} \rightarrow \Gamma_{n,q}$ by $\Psi_{n,q}(j) = \Psi_n(j + \ell_{n,q})$ for all $j \in \mathbb{Z}_{\ell_{n,q+1} - \ell_{n,q}}$. The resulting approximate slicing of $\{s_{p_n, j}^{(k_n)}\}_{\{r_{p_n} - k_n\}}$ are the dynamical sequences $\{s_{p_n, \Psi_n(q)(j)}^{(k_n)}\}_{\{\ell_{n,q+1} - \ell_{n,q}\}}$ indexed over q by $\{Q_n\}$.

Consider the following using the triangle inequality and Lemma 5.1 (ii):

$$\begin{aligned} & \sum_{i=0}^{h_n-1} |\mu(T^{t_n}(I_{p_n, i}) \cap B) - \mu(I_{p_n, i})\mu(B)| \\ &= \sum_{i=0}^{h_n-1} |\mu(T^{k_n h_{p_n} + m_n}(I_{p_n, i}) \cap B) - \mu(I_{p_n, i})\mu(B)| \\ &\leq \sum_{i=0}^{h_{p_n}-1} \left| \sum_{j=0}^{r_{p_n}-k_n-1} \mu(T^{-s_{p_n, j}^{(k_n)} + m_n}(I_{p_n, i}^{[j+k_n]}) \cap B) - \mu(I_{p_n, i}^{[j+k_n]})\mu(B) \right| \\ &\quad + \sum_{i=0}^{h_{p_n}-1} \sum_{j=r_{p_n}-k_n}^{r_{p_n}-1} |\mu(T^{k_n h_{p_n}}(I_{p_n, i}^{[j]}) \cap B) - \mu(I_{p_n, i}^{[j]})\mu(B)|. \end{aligned}$$

The ergodicity of $\{s_{n, j}\}_{\{r_n\}}$ with respect to T implies that the second summand above tends to zero (Theorem 2) using the same argument as above.

Note that

$$\begin{aligned} & \sum_{i=0}^{h_{p_n}-1} \left| \sum_{j=0}^{r_{p_n}-k_n-1} \mu(T^{-s_{p_n, j}^{(k_n)} + m_n}(I_{p_n, i}^{[j+k_n]}) \cap B) - \mu(I_{p_n, i}^{[j+k_n]})\mu(B) \right| \\ &= \sum_{i=0}^{h_{p_n}-1} \left| \sum_{q=0}^{Q_n-1} \sum_{j \in \Gamma_{n,q}} \mu(T^{-s_{p_n, j}^{(k_n)} + m_n}(I_{p_n, i}^{[j+k_n]}) \cap B) - \mu(I_{p_n, i}^{[j+k_n]})\mu(B) \right| \end{aligned}$$

Now for any $q \in \mathbb{Z}_{Q_n}$, any $i \in \mathbb{Z}_{h_{p_n}}$ and any $j \in \Gamma_{n,q}$, using Lemma 5.1 (ii),

$$\begin{aligned} & T^{-s_{p_n, j}^{(k_n)} + m_n}(I_{p_n, i}^{[j+k_n]}) \\ &= T^{-\left(s_{p_n, j}^{(k_n)} - s_{p_n, \Psi_n(\ell_{n,q})}^{(k_n)}\right) - s_{p_n, \Psi_n(\ell_{n,q})}^{(k_n)} + m_n}(I_{p_n, i}^{[j+k_n]}) \\ &= T^{-\left(s_{p_n, j}^{(k_n)} - s_{p_n, \Psi_n(\ell_{n,q})}^{(k_n)}\right) - s_{p_n, \Psi_n(\ell_{n,q})}^{(k_n)} + m_n + \alpha_{n,q} h_{p_n} + s_{p_n, j+k_n - \alpha_{n,q}}^{(\alpha_{n,q})}}(I_{p_n, i}^{[j+k_n - \alpha_{n,q}]}) \\ &= T^{-\left(s_{p_n, j}^{(k_n)} - s_{p_n, \Psi_n(\ell_{n,q})}^{(k_n)}\right) + \beta_{n,q}}(I_{p_n, i}^{[j+k_n - \alpha_{n,q}]}). \end{aligned}$$

If $i < h_{p_n} - \beta_{n,q}$ then

$$T^{-s_{p_n, j}^{(k_n)} + m_n}(I_{p_n, i}^{[j+k_n]}) = T^{-\left(s_{p_n, j}^{(k_n - \alpha_{n,q})} - s_{p_n, \Psi_n(\ell_{n,q})}^{(k_n - \alpha_{n,q})}\right)}(I_{p_n, i + \beta_{n,q}}^{[j+k_n - \alpha_{n,q}]}).$$

If $i \geq h_{p_n} - \beta'_{n,q}$ then

$$\begin{aligned}
& T^{-s_{p_n,j}^{(k_n)} + m_n}(I_{p_n,i}^{[j+k_n]}) \\
&= T^{-\left(s_{p_n,j}^{(k_n-\alpha_n,q)} - s_{p_n,\Psi_n(\ell_{n,q})}^{(k_n-\alpha_n,q)}\right) + \beta_{n,q} - h_{p_n} - s_{p_n,j+k_n-\alpha_n,q}}(I_{p_n,i}^{[j+k_n-\alpha_n,q+1]}) \\
&= T^{-\left(s_{p_n,j}^{(k_n-\alpha_n,q+1)} - s_{p_n,\Psi_n(\ell_{n,q})}^{(k_n-\alpha_n,q+1)}\right) + \beta_{n,q} - h_{p_n} - s_{p_n,\Psi_n(\ell_{n,q})+k_n-\alpha_n,q}}(I_{p_n,i}^{[j+k_n-\alpha_n,q+1]}) \\
&= T^{-\left(s_{p_n,j}^{(k_n-\alpha_n,q+1)} - s_{p_n,\Psi_n(\ell_{n,q})}^{(k_n-\alpha_n,q+1)}\right)}(I_{p_n,i+\beta'_{n,q}-h_{p_n}}^{[j+k_n-\alpha_n,q+1]})
\end{aligned}$$

as $i + \beta_{n,q} - h'_{p_n} \geq 0$ because $i \geq h_{p_n} - \beta'_{n,q}$ and $i + \beta'_{n,q} - h_{p_n} < i < h_{p_n}$ because $\beta'_{n,q} < h_{p_n}$.

If $h_{p_n} - \beta_{n,q} \leq i < h_{p_n} - \beta'_{n,q}$ then, as above,

$$\begin{aligned}
& T^{-s_{p_n,j}^{(k_n)} + m_n}(I_{p_n,i}^{[j+k_n]}) \\
&= T^{-\left(s_{p_n,j}^{(k_n-\alpha_n,q)} - s_{p_n,\Psi_n(\ell_{n,q})}^{(k_n-\alpha_n,q)}\right) + \beta_{n,q} - h_{p_n} - s_{p_n,j+k_n-\alpha_n,q}}(I_{p_n,i}^{[j+k_n-\alpha_n,q+1]}) \\
&= T^{-\left(s_{p_n,j}^{(k_n-\alpha_n,q+1)} - s_{p_n,\Psi_n(\ell_{n,q})}^{(k_n-\alpha_n,q+1)}\right)}(I_{p_n,i+\beta_{n,q}-h_{p_n}}^{[j+k_n-\alpha_n,q+1]}).
\end{aligned}$$

Applying the first case and Lemma 5.2, then that T is measure-preserving,

$$\begin{aligned}
& \sum_{i=0}^{h_{p_n}-\beta_{n,q}} \left| \sum_{q=0}^{Q_n-1} \sum_{j \in \Gamma_{n,q}} \mu(T^{-s_{p_n,j}^{(k_n)} + m_n}(I_{p_n,i}^{[j+k_n]}) \cap B) - \mu(I_{p_n,i}^{[j+k_n]})\mu(B) \right| \\
& \leq \int \left| \frac{1}{r_{p_n}} \sum_{q=0}^{Q_n-1} \sum_{j \in \Gamma_{n,q}} \chi_B \circ T^{-\left(s_{p_n,j}^{(k_n-\alpha_n,q)} - s_{p_n,\Psi_n(\ell_{n,q})}^{(k_n-\alpha_n,q)}\right)} - \mu(B) \right| d\mu \\
& \quad + \sum_{q=0}^{Q_n-1} \left(\sup_{j \in \Gamma_{n,q}} s_{p_n,j}^{(k_n-\alpha_n,q)} - s_{p_n,\Psi_n(\ell_{n,q})}^{(k_n-\alpha_n,q)} \right) \frac{1}{h_{p_n}} \frac{\#\Gamma_{n,q}}{r_{p_n}} \\
& \leq \int \left| \frac{1}{r_{p_n}} \sum_{q=0}^{Q_n-1} \sum_{j \in \Gamma_{n,q}} \chi_B \circ T^{-s_{p_n,j}^{(k_n-\alpha_n,q)}} - \mu(B) \right| d\mu + \epsilon_{p_n}.
\end{aligned}$$

Similarly, for the second and third cases above, we have that

$$\begin{aligned}
& \sum_{i=h_{p_n}-\beta'_{n,q}}^{h_{p_n}-1} \left| \sum_{q=0}^{Q_n-1} \sum_{j \in \Gamma_{n,q}} \mu(T^{-s_{p_n,j}^{(k_n)} + m_n}(I_{p_n,i}^{[j+k_n]}) \cap B) - \mu(I_{p_n,i}^{[j+k_n]})\mu(B) \right| \\
& \leq \int \left| \frac{1}{r_{p_n}} \sum_{q=0}^{Q_n-1} \sum_{j \in \Gamma_{n,q}} \chi_B \circ T^{-s_{p_n,j}^{(k_n-\alpha_n,q+1)}} - \mu(B) \right| d\mu + \epsilon_{p_n}
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{i=h_{p_n}-\beta_{n,q}}^{h_{p_n}-\beta'_{n,q}-1} \left| \sum_{q=0}^{Q_n-1} \sum_{j \in \Gamma_{n,q}} \mu(T^{-s_{p_n,j}^{(k_n)} + m_n}(I_{p_n,i}^{[j+k_n]}) \cap B) - \mu(I_{p_n,i}^{[j+k_n]})\mu(B) \right| \\
& \leq \int \left| \frac{1}{r_{p_n}} \sum_{q=0}^{Q_n-1} \sum_{j \in \Gamma_{n,q}} \chi_B \circ T^{-s_{p_n,j}^{(k_n-\alpha_n,q+1)}} - \mu(B) \right| d\mu + \epsilon_{p_n}
\end{aligned}$$

Combining these three cases, we have that

$$\begin{aligned}
& \sum_{i=0}^{h_{p_n}-1} \left| \sum_{q=0}^{Q_n-1} \sum_{j \in \Gamma_{n,q}} \mu(T^{-s_{p_n,j}^{(k_n)} + m_n}(I_{p_n,i}^{[j+k_n]}) \cap B) - \mu(I_{p_n,i}^{[j+k_n]})\mu(B) \right| \\
& \leq \int \left| \frac{1}{r_{p_n}} \sum_{q=0}^{Q_n-1} \sum_{j \in \Gamma_{n,q}} \chi_B \circ T^{-s_{p_n,j}^{(k_n - \alpha_n, q)}} - \mu(B) \right| d\mu \\
& \quad + 2 \int \left| \frac{1}{r_{p_n}} \sum_{q=0}^{Q_n-1} \sum_{j \in \Gamma_{n,q}} \chi_B \circ T^{-s_{p_n,j}^{(k_n - \alpha_n, q+1)}} - \mu(B) \right| d\mu + 3\epsilon_{p_n}
\end{aligned}$$

Note that $Q_n \leq \frac{s_{p_n, \Psi_n}(r_{p_n} - k_n - 1)}{\epsilon_{p_n} h_{p_n}} \leq \frac{s_{p_n, 0}^{(r_{p_n})}}{\epsilon_{p_n} h_{p_n}} = \frac{r_{p_n} \mu(S_{p_n})}{\epsilon_{p_n} \mu(C_{p_n})}$ so $\frac{Q_n}{r_{p_n}} \rightarrow 0$ as $n \rightarrow \infty$ by the construction of ϵ_n . Then the quantities above approach zero by slice-ergodicity around $\{k_n + 1\}$. \square

7. MIXING THEOREM

7.1. Mixing Rank-One Transformations. Our main theorem lifts the “restricted growth” condition from the main theorem (Theorem 6) of [CS04], generalizing it to all rank-one transformations.

Theorem 4. *For a rank-one transformation T , the following are equivalent:*

- (i) T is a mixing transformation;
- (ii) T is a rank-one uniform mixing transformation; and
- (iii) the spacer sequence for T is slice-ergodic (with respect to T).

Proof. Let T be as above. If (iii) holds, then Theorem 3 implies that every sequence is rank-one uniform mixing with respect to T so (ii) holds. If (ii) holds, then Proposition 5.1 implies that (i) holds. Assume that (i) holds but suppose that (iii) does not. Let $\{\Gamma_{n,q}\}$ define an approximate slicing not ergodic with respect to T . As $\frac{Q_n}{r_n} \rightarrow 0$, there exists a sequence $\{q_n\}$ such that $\#\Gamma_{n,q_n} \rightarrow \infty$ and so an approximate slice of the spacer sequence is monotone but not ergodic with respect to T . Theorem 1 then yields a contradiction. \square

8. POWER ERGODICITY

8.1. Power Ergodicity. Power ergodicity is all powers of an ergodic transformation being “uniformly” ergodic in the sense that the ergodic averages converge uniformly to zero. Earlier results on specific rank-one mixing used precursors to this notion, including the uniform Cesàro property used in [AF92] (and implicitly in [Ada98]) and power uniform ergodicity in [CS04].

Definition 8.1. *A transformation T is power ergodic when for all $B \in \mathcal{B}$,*

$$\lim_{n \rightarrow \infty} \sup_{k \in \mathbb{N}} \int \left| \frac{1}{n} \sum_{j=0}^{n-1} \chi_B \circ T^{-jk} - \mu(B) \right| d\mu = 0.$$

8.2. Weak Power Ergodicity.

Definition 8.2. A transformation T is **weak power ergodic** when for every sequence $\{k_n\}$ such that $\lim_{n \rightarrow \infty} \frac{k_n}{n} < \infty$ and all $B \in \mathcal{B}$,

$$\lim_{n \rightarrow \infty} \int \left| \frac{1}{n} \sum_{j=0}^{n-1} \chi_B \circ T^{-jk_n} - \mu(B) \right| d\mu = 0.$$

Weak power ergodicity was introduced in [CS04] as “power uniform ergodicity” however in light of the power ergodic property this name is more accurate.

Theorem 5. Let T be a rank-one transformation such that for each fixed $k \in \mathbb{N}$ the k^{th} partial sum sequence of the spacer sequence is ergodic with respect to T . Then T is weak power ergodic.

Proof. Let T be a rank-one transformation with spacer sequence $\{s_{n,j}\}_{\{r_n\}}$ and height sequence $\{h_n\}$ and let $k \in \mathbb{N}$. Note that $\frac{1}{h_n} s_{n,j}^{(k)} \rightarrow 0$ as $n \rightarrow \infty$ for a density one set of $j \in \mathbb{Z}_{r_n-k}$ because T is defined on a finite measure space (details are left to the reader). Applying Proposition 7.3 of [CS04] to those j (and ignoring the rest, a zero measure set) yields the conclusion. \square

8.3. Power Ergodicity Theorem.

Theorem 6. Let T be a rank-one transformation with spacers $\{s_{n,j}\}_{\{r_n\}}$ such that for any sequence $\{k_n\}$ where $\lim_{n \rightarrow \infty} \frac{k_n}{r_n} = 0$, the partial sum dynamical sequence $\{s_{n,j}^{(k_n)}\}_{\{r_n-k_n\}}$ is ergodic with respect to T . Then T is power ergodic.

Lemma 8.1. [Ada98] (**Block Lemma**) Let T be a measure-preserving transformation and $B \in \mathcal{B}$. Then for any $R, L, p \in \mathbb{N}$,

$$\int \left| \frac{1}{R} \sum_{j=0}^{R-1} \chi_B \circ T^{-j} - \mu(B) \right| d\mu \leq \int \left| \frac{1}{L} \sum_{j=0}^{L-1} \chi_B \circ T^{-jp} - \mu(B) \right| d\mu + \frac{pL}{R}.$$

Proof. (of Theorem 6) Let T be a rank-one transformation with spacer sequence $\{s_{n,j}\}_{\{r_n\}}$, let $\{k_n\}$ be an arbitrary sequence and $B \in \mathcal{B}$. For each $n \in \mathbb{N}$, set $p_n, q_n, x_n \in \mathbb{N}$ such that $h_{p_n} < k_n \leq h_{p_n+1}$, $h_{p_n+1} \leq k_n q_n < 2h_{p_n+1}$ and $x_n h_{p_n} \leq k_n < (x_n + 1)h_{p_n}$.

First, consider the case when $\frac{q_n}{n} \rightarrow 0$. Fix $\epsilon > 0$ and choose $L, N_0 \in \mathbb{N}$ such that $\frac{1}{L} < \epsilon$ and $\frac{q_n L}{n} < \epsilon$ for $n \geq N_0$. For each fixed $\ell < 2L$, the sequence $\{\ell h_{p_n+1}\}$ is mixing by Theorem 2 as $\{s_{n,j}^{(\ell)}\}_{\{r_n-\ell\}}$ is ergodic with respect to T by hypothesis. Since $\ell h_{p_n+1} \leq \ell k_n q_n < 2\ell h_{p_n+1}$, the sequence $\{\ell k_n q_n\}$ is then mixing (following from the construction of rank-one transformations). Let $N \in \mathbb{N}$ such that $|\mu(T^{\ell k_n q_n}(B) \cap B) - \mu(B)\mu(B)| < \epsilon$ for all $0 < \ell < L$. Then, for $n \geq \max(N_0, N)$, applying the Block Lemma (Lemma 8.1) and the Hölder Inequality,

$$\begin{aligned} & \int \left| \frac{1}{n} \sum_{j=0}^{n-1} \chi_B \circ T^{-jk_n} - \mu(B) \right| d\mu \\ & \leq \int \left| \frac{1}{L} \sum_{\ell=0}^{L-1} \chi_B \circ T^{-\ell k_n q_n} - \mu(B) \right| d\mu + \frac{q_n L}{n} \\ & \leq \left[\frac{1}{L} \sum_{\ell=-L+1}^{L-1} \frac{L-\ell}{L} (\mu(T^{\ell k_n q_n}(B) \cap B) - \mu(B)\mu(B)) \right]^{\frac{1}{2}} + \epsilon < 2\epsilon. \end{aligned}$$

Now, consider the case when $\frac{q_n}{n} \geq \delta$ for some $\delta > 0$. Since $\frac{q_n}{n} < \frac{2h_{p_n+1}}{nk_n} \leq \frac{2h_{p_n+1}}{nx_n h_{p_n}} \approx \frac{2r_{p_n}}{nx_n}$ (by finite measure-preserving), $\frac{x_n}{r_{p_n}} < \frac{1}{\delta n}$. Choose a sequence $\{L_n\}$ such that $L_n \rightarrow \infty$ and $\frac{L_n}{n} \rightarrow 0$. For any sequence $\{\ell_n\}$ such that $\ell_n < L_n$, we see that $\frac{\ell_n x_n}{r_{p_n}} < \frac{L_n}{n\delta} \rightarrow 0$. By hypothesis, this means that $\{s_{p_n, j}^{(\ell_n x_n)}\}_{\{r_{p_n} - \ell_n x_n\}}$ is ergodic with respect to T . Theorem 5 of [CS04] then yields that $\{\ell_n k_n\}$ is a mixing sequence. Applying the Block Lemma (Lemma 8.1) and the Hölder Inequality,

$$\begin{aligned} & \int \left| \frac{1}{n} \sum_{j=0}^{n-1} \chi_B \circ T^{-jk_n} - \mu(B) \right| d\mu \\ & \leq \int \left| \frac{1}{L_n} \sum_{\ell=0}^{L_n-1} \chi_B \circ T^{-\ell k_n} - \mu(B) \right| d\mu + \frac{L_n}{n} \\ & \leq \left[\frac{1}{L_n} \sum_{\ell=-L_n+1}^{L_n-1} \frac{L_n - \ell}{L_n} (\mu(T^{\ell k_n}(B) \cap B) - \mu(B)\mu(B)) \right]^{\frac{1}{2}} + \frac{L_n}{n} \rightarrow 0. \end{aligned}$$

□

8.4. Polynomial Power Ergodicity Polynomial power ergodicity is the “polynomial powers” of a transformation being “uniformly ergodic”. The term **polynomial** shall mean polynomials with rational coefficients that map integers to integers.

Definition 8.3. A transformation T is **polynomial power ergodic** when for all sequences of polynomials $\{p_n\}$ of bounded degree and all $B \in \mathcal{B}$,

$$\lim_{n \rightarrow \infty} \int \left| \frac{1}{n} \sum_{j=0}^{n-1} \chi_B \circ T^{-p_n(j)} - \mu(B) \right| d\mu = 0;$$

T is **weak polynomial power ergodic** when the polynomial power ergodicity condition holds for polynomial sequences $\{p_n\}$ of bounded degree such that $\lim_{n \rightarrow \infty} \frac{c_n}{n} < \infty$ where $\{c_n\}$ are the lead coefficients of the $\{p_n\}$.

Theorem 7. [CS04] Let T be a transformation that is weak power ergodic. Then T is weak polynomial power ergodic.

Theorem 8. Let T be a transformation that is power ergodic. Then T is polynomial power ergodic.

Proof. Identical to that of Theorem 8 in [CS04] (which is Theorem 7 above) omitting the condition for weakness. □

9. MIXING RANK-ONE TRANSFORMATIONS

9.1. Staircase Transformations. Let $\{r_n\}$ be a sequence and T a rank-one transformation with spacer sequence $\{s_{n,j}\}_{\{r_n\}}$ given by $s_{n,j} = j$. Then T is a **staircase transformation**.

9.2. Polynomial Staircase Transformations. Let $\{p_n\}$ be a sequence of polynomials with bounded degree. A rank-one transformation with spacer sequence

$\{s_{n,j}\}_{\{r_n\}}$ given by $s_{n,j} = p_n(j)$ is a **polynomial staircase transformation**. We require that the polynomials be such that for every $L \in \mathbb{N}$,

$$\limsup_{n \rightarrow \infty} \frac{1}{r_n} \#\{j \in \mathbb{Z}_{r_n} : L \text{ divides } p_n(j+1) - p_n(j)\} < 1$$

and such that $\lim_{n \rightarrow \infty} \frac{c_n}{n} < \infty$, where $\{c_n\}$ are the lead coefficients of the polynomials.

Remark. Adams and Friedman in [AF92] introduced polynomial staircase transformations as models for rank-one mixing transformations. A class of mixing polynomial staircase transformations was studied in the first author's undergraduate thesis at Williams College in 2001. Polynomial staircase transformations with restricted growth were shown to be mixing in [CS04]. After [CS04] was accepted, Ryzhikov wrote the authors that a similar class of regular behavior polynomial staircases was shown to be mixing by him in 2002, but a preprint was not available [Rhy03].

The following theorem is essentially the polynomial ergodic theorem of Furstenberg [Fur81, p. 70] applied to a sequence of polynomials with uniformly bounded coefficients (see [CS04] for a short proof).

Theorem 9. [Furstenberg] *Let $\{p_n\}$ be a sequence of polynomials of bounded degree and uniformly bounded coefficients. Then a dynamical sequence $\{s_{n,j}\}_{\{r_n\}}$ given by $s_{n,j} = p_n(j)$ is ergodic with respect to any totally ergodic transformation.*

Theorem 10. *Polynomial staircase transformations are mixing.*

Proof. Let T be a polynomial staircase transformation. Then T is totally ergodic by Theorem 7 of [CS04]. Let $\{p_n\}$ be the polynomials defining the spacer sequence $\{s_{n,j}\}_{\{r_n\}}$ of degree at most $D \in \mathbb{N}$ and let $\{c_{n,a}\}$ for $a \in \mathbb{Z}_{D+1}$ be the coefficients of the $\{p_n\}$. Then, for any $j \in \mathbb{Z}_{r_n}, k \in \mathbb{Z}_{r_n-j}$,

$$\begin{aligned} s_{n,j}^{(k)} &= \sum_{z=0}^{k-1} p_n(j+z) = \sum_{z=0}^{k-1} \sum_{a=0}^D c_{n,a} (j+z)^a = \sum_{z=0}^{k-1} \sum_{a=0}^D \sum_{b=0}^a c_{n,a} \binom{a}{b} j^b z^{a-b} \\ &= \sum_{b=0}^D \left(\sum_{z=0}^{k-1} \sum_{a=b}^D c_{n,a} \binom{a}{b} z^{a-b} \right) j^b = p_{n,k}(j) \end{aligned}$$

are polynomials of degree at most D in j with lead coefficients $kc_{n,D}$.

T is weak power ergodic by Theorem 5 as for each fixed $k \in \mathbb{N}$ the $p_{n,k}$ are ergodic (Theorem 9) since the coefficients of $p_{n,k}$ are uniformly bounded when k is fixed. T is then weak polynomial power ergodic by Theorem 7. For any sequence $\{k_n\}$ such that $\frac{k_n}{r_n} \rightarrow 0$, the partial sum sequence $\{s_{n,j}^{(k_n)}\}_{\{r_n-k_n\}}$ is ergodic with respect to T by weak polynomial power ergodicity (as $s_{n,j}^{(k_n)} = p_{n,k_n}(j)$ where p_{n,k_n} have lead coefficients $k_n c_n$ and $\frac{k_n c_n}{n} \rightarrow 0$ as $\lim_{n \rightarrow \infty} \frac{c_n}{n} < \infty$). Hence, T is power ergodic by Theorem 6 and so is polynomial power ergodic by Theorem 8.

For any $\{k_n\}$ and approximate slicing $\{\Gamma_{n,q}\}, \{\Psi_{n,q}\}, \{\alpha_{n,q}\}_{\{Q_n\}}$ of the spacer sequence around $\{k_n\}$, each slice $\{s_{n,\Psi_{n,q}(j)}^{(k_n-\alpha_{n,q})}\}_{\{\#\Gamma_{n,q}\}}$ is itself a polynomial sequence (details are left to the reader) of degree at most D . Since $\frac{Q_n}{r_n} \rightarrow 0$, $\#\Gamma_{n,q} \rightarrow \infty$ uniformly over a density one set of q , the slices then uniformly tend to zero by polynomial power ergodicity. Theorem 4 then yields the result. \square

As mentioned in the introduction, Ryzhikov wrote the authors that he had proved the following Corollary 11 [Rhy03]; it answers the question asked by Adams in [Ada98].

Corollary 11. *Staircase transformations are mixing.*

10. SPECIFIC MIXING TRANSFORMATIONS

10.1. Criterion for Finite Measure on Rank-One Transformations

Proposition 10.1. *A rank-one transformation with spacer sequence $\{s_{n,j}\}_{\{r_n\}}$ and heights $\{h_n\}$ is defined on a finite measure space if and only if*

$$\sum_{n=0}^{\infty} \frac{\bar{s}_n}{h_n} < \infty \quad (\text{where } \bar{s}_n = \frac{1}{r_n} \sum_{j=0}^{r_n-1} s_{n,j}).$$

Proof. Let T , $\{s_{n,j}\}_{\{r_n\}}$, $\{\bar{s}_n\}$, and $\{h_n\}$ be as above and let (X, μ) be the space T is defined on. Let $\{C_n\}$ denote the columns of the construction as sets, $\{I_n\}$ the base levels of the columns, and $\{S_n\}$ the spacers added (so $S_n = C_{n+1} \setminus C_n$). We see that

$$\mu(S_n) = \sum_{j=0}^{r_n-1} s_{n,j} \mu(I_{n+1}) = \left(\frac{1}{r_n} \sum_{j=0}^{r_n-1} s_{n,j} \right) \mu(I_n) = \frac{\bar{s}_n}{h_n} \mu(C_n)$$

and so

$$\frac{\mu(C_{n+1})}{\mu(C_n)} = \frac{\mu(C_n) + \mu(S_n)}{\mu(C_n)} = 1 + \frac{\bar{s}_n}{h_n}.$$

Then,

$$\log \left(\frac{\mu(X)}{\mu(C_0)} \right) = \log \left(\prod_{n=0}^{\infty} \frac{\mu(C_{n+1})}{\mu(C_n)} \right) = \sum_{n=0}^{\infty} \log \left(1 + \frac{\bar{s}_n}{h_n} \right) \approx \sum_{n=0}^{\infty} \frac{\bar{s}_n}{h_n}$$

using the approximation $\log(1 + \epsilon) \approx \epsilon$ for small ϵ .

Since $\mu(X) < \infty$ if and only if $\frac{\mu(X)}{\mu(C_0)} < \infty$, the result follows. \square

10.2. Specific Examples of Mixing Transformations.

Definition 10.1. *Let $D \in \mathbb{N}$ and $\delta \in \mathbb{R}^+$. The rank-one transformation $T_{D,\delta}$ with spacers $\{s_{n,j}\}_{\{r_n\}}$ given by $s_{n,j} = j^D$ and $r_n = \lfloor h_n^{\frac{1}{D+\delta}} \rfloor$ (where $\{h_n\}$ is the heights for $T_{D,\delta}$) is a **simple polynomial staircase transformation**.*

Theorem 12. *Simple polynomial staircase transformations are mixing.*

Proof. By Theorem 10, we need only show the transformations are defined on a finite measure space. Let $D \in \mathbb{N}$ and $\delta \in \mathbb{R}^+$. Let $\{s_{n,j}\}_{\{r_n\}}$ be the spacers and $\{h_n\}$ the heights for $T_{D,\delta}$. Now, $\sum_{j=0}^{r_n-1} s_{n,j} = \sum_{j=0}^{r_n-1} j^D \approx r_n^{D+1}$. Since $h_{n+1} = r_n h_n + \sum_{j=0}^{r_n-1} s_{n,j}$, we see that $h_n \geq \prod_{z=0}^{n-1} r_z \geq 2^n$. Then,

$$\sum_{n=0}^{\infty} \frac{\bar{s}_n}{h_n} \approx \sum_{n=0}^{\infty} \frac{1}{r_n h_n} r_n^{D+1} = \sum_{n=0}^{\infty} \frac{r_n^D}{h_n} \approx \sum_{n=0}^{\infty} h_n^{\frac{D}{D+\delta}-1} \leq \sum_{n=0}^{\infty} (2^{\frac{\delta}{D+\delta}})^{-n} < \infty$$

by the convergence of geometric series. Proposition 10.1 completes the proof. \square

10.3. Ornstein's Transformation. Ornstein's original construction of rank-one mixing transformations involved placing spacer levels randomly using a uniform distribution so that almost surely the resulting transformation is mixing. The uniform distribution can be equally well interpreted as meaning that the spacer sequence is almost surely slice-ergodic so that mixing for these transformations follows from our theorem. The reader is referred to [CS04] for details.

10.4. A Note on Restricted Growth. The restricted growth condition in [CS04] is equivalent to $\frac{r_n \bar{s}_n}{h_n} \rightarrow 0$ for polynomial staircase transformations (details are left to the reader). For $T_{D,\delta}$, we see that $\frac{r_n \bar{s}_n}{h_n} \approx \frac{r_n^{D+1}}{h_n} \approx h_n^{\frac{D+1}{D+\delta}-1} = h_n^{\frac{1-\delta}{D+\delta}}$ and so $T_{D,\delta}$ has restricted growth if and only if $\delta > 1$. Hence, the theorems of [CS04] apply only to $T_{D,\delta}$ with $\delta > 1$. The $T_{D,\delta}$ for $0 < \delta \leq 1$ are not provably mixing using previous results. In particular, the staircase transformations $T_{1,\delta}$ for $0 < \delta \leq 1$ are mixing (the case $\delta > 1$ was first shown by Adams in [Ada98]; another proof was given in [CS04]).

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